TRANSIENT THERMAL STRESSES IN VISCOELASTIC PLATES AND SHELLS

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1. INTRODUCTION: HYPOTHESES, GENERAL RELATIONS AND EQUATIONS

The assumption of perfect elasticity in investigations on thermoelastic behaviour of various solids limits the range of applicability of the solutions to comparatively small range of temperature. Under higher temperatures the solid exhibits properties of creep and the stresses calculated on the basis of the theory of elasticity differ from the observed, the difference depending on time, and for high temperatures also on the temperature.

In this paper we shall deal with discs, plates and shells made of a viscoelastic material possessing a linear characteristic, which constitutes a better approximation to the behaviour of solids subject to temperature fields. We assume that the solid is homogeneous and isotropic. Moreover, we confine ourselves to small deformations and we make the assumption that all physical properties of the material are independent of temperature and the yield limit is not reached. The assumption of independence of the physical properties of temperature is a significant limitation, for it contradicts the familiar influence of the temperature on the viscosity modulus. Though taking into account the latter phenomenon is possible, it leads to considerable mathematical difficulties.

In viscoelastic solids the transform properties of the stress and strain tensors are the same as in the case of perfectly elastic solids. The differences arise only in the stress-strain laws[1,2,6]

$$P_1(D)s_{ij} = P_2(D)e_{ij} \tag{1.1}$$

$$P_3(D)\sigma_{kk} = P_4(D)\,(\varepsilon_{kk}-3\alpha_t T) \tag{1.2}$$

where

$$s_{ij} = \sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}, \quad e_{ij} = \varepsilon_{ij}-\frac{1}{3}\varepsilon_{kk}\delta_{ij} \tag{1.3}$$

are the deviators of the stress and strain tensors, respectively, and T denotes the temperature. The quantities $P_i(D)$, $i = 1,2,3,4$ appearing in the relations (1.1) and (1.2) constitute linear differential operators

$$P_i(D) = \sum_{n=0}^{Ni} a_i^{(n)} D^n, \quad a_i^{(Ni)} \neq 0, \quad i = 1,2,3,4 \tag{1.4}$$

where $D^n = \dfrac{\partial^n}{\partial t^n}$ denotes the derivative of order n with respect to time, and the coefficients $a_i^{(n)}$ are constants. In the case of a perfectly elastic body the operators $P_i(D)$ contain only the first term of the series (1.4). Namely we have

$$a_1^{(0)} = 1, \quad a_2^{(0)} = 2\mu_0, \quad a_3^{(0)} = 1, \quad a_4^{(0)} = 3K_0$$

where μ_0 is the shear modulus and K_0 the compression modulus.

The system of relations (1.1)–(1.2) can be written in a more explict form

$$P_1(D)P_3(D)\sigma_{ij} = P_2(D)P_3(D)\varepsilon_{ij} + \\ + \delta_{ij}\left\{\frac{1}{3}[P_1(D)P_4(D) - P_2(D)P_3(D)]\varepsilon_{kk} - P_1(D)P_4(D)\alpha_t T\right\} \tag{1.5}$$

Besides the model of the viscoelastic solid described by the relations (1.5) frequently models of continuous spectrum are used, namely the Boltzmann and Biot models[3,4]

$$\sigma_{ij} = 2\int_0^t a(t-\tau)\frac{\partial \varepsilon_{ij}}{\partial \tau}\,d\tau + \\ + \delta_{ij}\int_0^t \left\{b(t-\tau)\frac{\partial \varepsilon_{KK}}{\partial \tau} - [3b(t-\tau) + 2a(t-\tau)]a_t\frac{\partial T}{\partial \tau}\right\}d\tau \tag{1.6}$$

This relation defines a viscoelastic solid free of stresses at the initial instant. $a(t)$, $b(t)$ are here functions of time which for the perfectly elastic solid reduce to the Lamé constants μ_0, λ_0.

Let us perform in the relations (1.5) and (1.6) the Laplace transformation defined as follows:

$$\bar{f}(x_r, p) = \int_0^\infty e^{-pt} f(x_r, t)\,dt$$

Thus we arrive at the relations

$$\bar{\sigma}_{ij} = 2\bar{\mu}\bar{\varepsilon}_{ij} + (\bar{\lambda}\bar{\varepsilon}_{KK} - \bar{T}\bar{\gamma})\,\delta_{ij} \qquad ij = 1,2,3 \tag{1.7}$$

where

$$\bar{\mu} = \frac{P_2(p)}{2P_1(p)}, \quad \bar{\lambda}(p) = \frac{P_1(p)P_4(p) - P_2(p)P_3(p)}{3P_1(p)P_3(p)}, \quad \bar{\gamma} = (2\bar{\mu} + 3\bar{\lambda})\alpha_t \tag{1.8}$$

for the stress-strain relation expressed by (1.5), and

$$\bar{\mu} = pa(p), \quad \bar{\lambda} = pb(p) \tag{1.9}$$

for the Biot model of viscoelastic solid. The corresponding relations for a perfectly elastic solid have the form

$$\bar{\sigma}_{ij}^{(0)} = 2\mu_0 \bar{\varepsilon}_{ij}^{(0)} + (\lambda_0 \bar{\varepsilon}_{KK}^{(0)} - \gamma_0 \bar{T})\,\delta_{ij} \tag{1.10}$$

Consider first the quasi-static problem. In other words we assume that the temperature changes are slow and therefore we neglect the influence of the inertia forces on strains and stresses. Substituting into the equilibrium equations

$$\sigma_{ij,j} = 0 \tag{1.11}$$

the stresses in accordance with the relations (1.5) and expressing the strains by the displacements by means of the formula

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.12}$$

we arrive after simple transformations at the displacement equations

$$L_1(D)u_{i,KK} + L_2(D)u_{K,Ki} = L_3(D)\alpha_t T_{,i} \qquad i = 1,2,3 \tag{1.13}$$

where

$$L_1(D) = P_2(D)P_3(D), \qquad L_2(D) = \frac{1}{3}[2P_4(D)P_1(D) + P_2(D)P_3(D)],$$
$$L_3(D) = 2P_4(D)P_1(D)$$

The equations (1.13) are to be completed by boundary conditions

$$\sigma_{ij} n_j = p_i \tag{1.14}$$

in the case of loading of the boundary, and

$$u_i = g_i(x_r, t) \tag{1.15}$$

in the case of kinematic boundary conditions. We perform the Laplace transform on the displacement equations (1.13) and on the boundary conditions, assuming that for $t \leqslant 0$ the viscoelastic solid is in natural state, i.e. the displacement and stresses vanish. In other words we assume that $u_i(x_r, 0) = \dot{u}_i(x_r, 0) = 0$, and the higher derivatives defined by the order of the operator $P_i(D)$. also vanish. Thus we arrive at the system of equations

$$\bar{\mu}\bar{u}_{i,KK} + (\bar{\lambda} + \bar{\mu})\bar{u}_{K,Ki} = \bar{\gamma}\bar{T}_{,i} \qquad i = 1,2,3 \tag{1.16}$$

the corresponding equations for a perfectly elastic solid having the form

$$\mu_0 \bar{u}_{i,KK}^{(0)} + (\lambda_0 + \mu_0)\bar{u}_{K,Ki}^{(0)} = \gamma_0 \bar{T}_{,i} \tag{1.17}$$

A comparison of the equations (1.16) and (1.17) indicates that the equations for the viscoelastic solid are derived by replacing the constants μ_0,

λ_0 in the displacement equations of the perfectly elastic solid, by the quantities $\overline{\mu}$, $\overline{\lambda}$ which depend on the parameter of the Laplace transformation p. This analogy is called the elastic-viscoelastic analogy. The following procedure for solving the displacement equations is evident. We solve the quasi-static equations for the perfectly elastic solid and in the solution thus obtained we replace the constants μ_0, λ_0 by $\overline{\mu}$, $\overline{\lambda}$ and apply the inverse Laplace transformation; this completes the solution of the corresponding viscoelastic problem, in the quasi-static case.

Owing to the elastic-viscoelastic analogy we can use numerous solutions of thermoelasticity to obtain the solutions of the corresponding viscoelastic problems. This analogy was announced by Alfrey[5] who employed it in determining the stresses in a viscoelastic solid, produced by external loading. The analogy was extended by Lee[6] to the viscoelastic bodies in which the condition of incompressibility is not valid. Alfrey's analogy was first made use of in thermoelasticity by Hilton[7], and Sternberg who disregarded the condition of incompressibility[2].

The solution of the system of equations (1.16) can be represented in the form[8]

$$\overline{u}_i(x_r, p) = \overline{\gamma} \int_{(V)} \overline{T}(\xi_r, p)\overline{\theta}_i(\xi_r, x_r, p)\,\mathrm{d}V \qquad i = 1,2,3 \tag{1.18}$$

Here $\theta_i(\xi_r, x_r, p)$ denotes the Laplace transformation of the dilatation at a point (ξ_r), produced by an instantaneous concentrated force acting at a point (x_r) in the direction of the x_i-axis, in a viscoelastic solid of the same shape as the solid under consideration, under the assumption that $T = 0$.

The corresponding solution of the system of equations (1.17) for the perfectly elastic solid has the form

$$\overline{u}_i^{(0)}(x_r, p) = p_0 \int_{(V)} \overline{T}(\xi_r, p)\Theta_i^{(0)}(\xi_r, x_r)\,\mathrm{d}V \qquad i = 1,2,3 \tag{1.19}$$

Here $\Theta_i^{(0)}(\xi_r, x_r)$ is the sum of normal stresses at a point (ξ_r) of the perfectly elastic body, produced by a concentrated independent of time force acting in the direction of the x_i-axis at the point (x_r).

Differentiating (1.18) we obtain

$$\overline{\varepsilon}_{ij}(x_r, p) = \tfrac{1}{2}(\overline{u}_{i,j} + \overline{u}_{j,i}) = \overline{\gamma} \int_{v} \overline{T}(\xi_r, p)\overline{\Theta}_{ij}(\xi_r, x_r, p)\,\mathrm{d}V \tag{1.19}$$

Here, for $i \neq j$ the function Θ_{ij} denotes the dilatation at the point (ξ_r), due to the action of an instantaneous cross of shear situated at the point (x_r) of the plane $x_i x_j$. When $i = j$ the function Θ_{ii} is the dilatation at the point (ξ_r), due to the action of an instantaneous double force applied to the point (x_r) in the direction of the x_i-axis.

The transformation of the dilatation $e(x_r, t)$ at the point (x_r) of a viscoelastic body, due to the action of a temperature field, is given by the formula

$$\bar{e}(x_r, p) = \bar{u}_{i,i} = \bar{\gamma} \int_v \bar{T}(\xi_r, p)\bar{\Theta}(\xi_n, x_n, p)\,dV \tag{1.20}$$

$$= \alpha_t \int_v \bar{T}(\xi_n, p)\bar{\Lambda}(\xi_r, x_r, p)\,dV$$

Here $\Theta(\xi_r, x_r, t)$ denotes the dilatation at the point (ξ_r), due to the action of an instantaneous centre of pressure at the point (x_r); $\Lambda(\xi_r, x_r, t)$ is the sum of the normal stresses at the point (ξ_r), due to the action of an instantaneous centre of pressure at the point (x_r).

By means of the above method of solution certain general results may be obtained.

First, we shall prove that the change of volume of a viscoelastic body free of tractions on its surface, due to the existence of a temperature field, is independent of the rheological properties of the body.

The change of volume is given by the integral

$$\Delta\bar{V} = \int_{v_1} \bar{e}(x_r, p)\,dV_1 = \alpha_t \int_v \bar{T}(\xi_r, p)\,dV \int_{v_1} \bar{\Lambda}(\xi_r, x_r, p)\,dV_1, \tag{1.21}$$

$$dV_1 = dx_1, dx_2, dx_3$$

The second integral constitutes the sum of stresses at the point (ξ_r), due to the action of centres of pressure distributed uniformly over the volume of the body, and is equal to 3. Hence

$$\Delta V = 3\alpha_t \int_v T(\xi_r, t)\,dV \tag{1.22}$$

Taking into account that

$$\bar{e} = \frac{\bar{s}}{3\bar{\lambda}+2\bar{\mu}} + 3\alpha_t \bar{T}, \qquad \bar{s} = \bar{\sigma}_{kk} \tag{1.23}$$

and making use of the relations (1.21) and (1.22) we have

$$\int_{v_1} s(x_r, t)\,dV_1 = 0 \tag{1.24}$$

The change of volume due to the thermal stresses vanishes; there remains only the change due to the temperature. It is easy to verify the validity of the relations

$$\int_{v_1} \varepsilon_{ij}(x_r, t)\,dV_1 = \alpha_t \delta_{ij} \int_{v_1} T(x_r, t)\,dV_1 \tag{1.25}$$

Let us next consider a viscoelastic body with vanishing displacements on the surface. Assuming that the temperature T_0 is constant inside the body we arrive at the following formulae for the deformations:

$$\bar{\varepsilon}_{ij}(x_r, p) = \frac{T_0 \bar{\gamma}}{p} \int_v \bar{\Theta}_{ij} \, \mathrm{d}V \tag{1.26}$$

These deformations vanish, since the integral on the right-hand side denotes the increment of the volume of the body, due to the action of the uniformly distributed crosses of shear ($i \neq j$) or double forces ($i = j$). It follows from the formula (1.7) that

$$\bar{\sigma}_{ij} = -\frac{T_0}{p} \delta_{ij} \bar{\gamma} \tag{1.27}$$

The determination of stresses in an infinite viscoelastic space is especially simple.

We now prove that the following simple relations occur between the displacements $u_i(x_r, t)$ and the displacements $u_i^{(0)}(x_r, t)$ of the viscoelastic body[8]:

$$\bar{u}_i(x_r, p) = \bar{f}(p) p u_i^{(0)}(x_r, p), \quad \bar{f}(p) = \frac{\bar{\gamma}(\lambda_0 + 2\mu_0)}{\gamma_0 p(\bar{\lambda} + 2\bar{\mu})} \tag{1.28}$$

and

$$u_i(x_r, t) = \int_0^t f(t-\tau) \frac{\partial}{\partial \tau} u_i^{(0)}(x_r, \tau) \, \mathrm{d}\tau \tag{1.29}$$

where the function $f(t)$ describes the rheological properties of the viscoelastic body.

Let us observe that the relations (1.18) and (1.19) can be represented in the form[8]

$$\bar{u}_i(x_r, p) = -\bar{\gamma} \int_v \bar{T}(\xi_r, p) \bar{U}_i(x_r, \xi_r, p) \, \mathrm{d}V \tag{1.130}$$

$$\bar{u}_i^{(0)}(x_r, p) = -\gamma_0 \int_v T(\xi_r, p) U_i^{(0)}(x_r, \xi_r) \, \mathrm{d}V \tag{1.31}$$

Here $U_i(x_r, \xi_r, t)$ denotes the displacement of the point (x_r) in the direction of the x_i-axis, due to the action of an instantaneous centre of pressure situated at the point (ξ_r). It is known from the theory of elasticity that

$$U_i^{(0)}(x_r, \xi_r) = -\frac{1}{4\pi\beta_0} \left(\frac{1}{R}\right)_{,i}$$

$$\beta_0 = \lambda_0 + 2\mu_0$$

where

$$R = [(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2$$

In accordance with the viscoelastic analogy we have

$$\overline{U}_i(x_r, \xi_r, p) = -\frac{1}{4\pi\bar{\beta}}\left(\frac{1}{R}\right)_{,i}$$

$$\bar{\beta} = \bar{\lambda}+2\bar{\mu}.$$

The relations (1.28) will be called the correspondence relations principle. They enable us to determine the displacements after carrying out the convolution 1.29, if the displacements $u_i^{(0)}$ and the function $f(t)$ describing viscoelastic properties of the solid are known. Making use of the correspondence relations (1.28) and the equations (1.7) we represent the thermal stresses in a viscoelastic solid by the relation

$$\sigma_{ij}(\chi_r, t) = \int_0^t \left\{2h(t-\tau)\frac{\partial \varepsilon_{ij}^{(0)}}{\partial \tau}+\delta_{ij}\left[k(t-\tau)\frac{\partial \varepsilon_{kk}^{(0)}}{\partial \tau}\, l(t-\tau)\frac{\partial T}{\partial \tau}\right]\right\} \mathrm{d}t \qquad (1.32)$$

where

$$h(t) = \alpha^{-1}(\bar{\mu}\bar{f}), \quad k(t) = \alpha^{-1}(\bar{\lambda}\bar{f}), \quad \bar{l} = \alpha^{-1}(\bar{\gamma}\bar{f})$$

The validity of the relations (1.28) follows from the comparison of the formulae (1.30) and (1.31). Moreover, this comparison indicates that the transform of the function $\bar{u}_i$ can be expressed in terms of the function ϕ as follows:

$$\bar{u}_i = \bar{\phi}_{,i} \qquad (1.33)$$

where

$$\bar{\phi} = -\frac{\bar{m}}{4\pi}\int_v \frac{\overline{T}(\xi_r, p)\,\mathrm{d}V}{R}, \quad \bar{m} = \frac{\bar{\gamma}}{\bar{\beta}} \qquad (1.34)$$

On the other hand we find that (1.34) is a solution of the Poisson equation

$$\bar{\phi}_{,kk} = \bar{m}\overline{T}. \qquad (1.35)$$

In fact, introducing (1.33) into the displacement equations (1.16) and integrating with respect to x_i we arrive at the equation (1.35). The function ϕ is the so-called thermoelastic deformation potential. In terms of this function the stresses $\bar{\sigma}_{ij}$ can be expressed as follows:

$$\bar{\sigma}_{ij} = 2\bar{\mu}(\bar{\phi}_{,ij}-\delta_{ij}\phi_{,kk}), \quad i,j = 1, 2, 3 \qquad (1.36)$$

Comparing the equation (1.35) with the corresponding equation for the viscoelastic body find that

$$\phi(x_r, t) = \int_0^t g(t-\tau)\frac{\partial}{\partial \tau}-\phi^{(0)}\,\mathrm{d}\tau, \qquad (1.37)$$

where

$$\bar{g}(p) = \frac{\bar{m}}{pm_0}$$

The stresses σ_{ij} in the viscoelastic body are given in terms of the function $\phi^{(0)}$ for the linear-elastic body by the relation

$$\sigma_{ij}(x_r, t) = 2[\partial_i\partial_j - \delta_{ij}\nabla^2] \int_0^t r(t-\tau)\frac{\partial\phi^{(0)}}{\partial\tau}\,\mathrm{d}\tau, \tag{1.38}$$

where

$$\bar{r}(p) = \frac{\bar{\mu}\bar{m}}{pm_0}$$

If the changes in time of the temperature field are not sufficiently slow the inertia forces cannot be neglected. In this case the equations of motion have the form

$$\sigma_{ij,j} = \varrho\ddot{u}_i \qquad i, j = 1, 2, 3 \tag{1.39}$$

Introducing into these equations the relations (1.5) we obtain the dynamical equations

$$L_1(D)u_{i,KK} + L_2(D)u_{K,Ki} = L_3(D)\alpha_t T_{,i} + L_4(D)\varrho\ddot{u}_i, \qquad i = 1, 2, 3 \tag{1.40}$$

where

$$L_4(D) = 2P_1(D)P_3(D)$$

Applying the Laplace transformation to the equations (1.40) and comparing it with the appropriate equation for a perfectly elastic solid we discover that in this case also the elastic-viscoelastic analogy holds. It is however impossible to establish any correspondence relations as it was done in quasi-static problems.

In an infinite viscoelastic solid, introducing the potential ϕ we reduce the system of equations (1.40) to the equation (10)

$$\bar{\phi}_{,KK} - p^2\bar{\sigma}^2(p)\bar{\phi} = \bar{n}\bar{T} \tag{1.41}$$

where

$$\bar{\sigma}^2(p) = \frac{\varrho}{\bar{\lambda}+2\bar{\mu}}$$

The transformations of the stresses assume in this case the form

$$\bar{\sigma}_{ij} = 2\bar{\mu}(\partial_i\partial_j - \delta_{ij}\nabla^2)\bar{\phi} + p^2\varrho\bar{\phi} \tag{1.42}$$

The corresponding equation for the function $\bar{\phi}^{(0)}$ in the case of a perfectly elastic body is the following:

$$\bar{\phi}^{(0)}_{,KK} - p^2\sigma_0^2\bar{\phi}^{(0)} = n_0\bar{T} \tag{1.43}$$

$$\sigma_0^2 = \frac{\varrho}{\lambda_0+2\mu_0}$$

The transformations of the stresses have a similar form

$$\overline{\sigma}_{ij}^{(0)} = 2\mu_0(\partial_i\partial_j - \delta_{ij}\nabla^2)\overline{\phi}^{(0)} + p^2\varrho\overline{\phi}^{(0)} \tag{1.44}$$

It follows from a comparison of the equations (1.41) and (1.43) that no correspondence relations are valid in the case of the dynamical problem, for the functions $\overline{\phi}$ and $\overline{\phi}^{(0)}$.

2. THERMAL STRESSES IN DISCS

Consider a disc of constant thickness h situated in a non-stationary temperature field $T(x_1, x_2, t)$ satisfying the heat conduction equation[11]

$$\nabla_1^2 T - \frac{1}{\varkappa}\dot{T} - \varepsilon(T-\theta) = -Q/\varkappa \tag{2.1}$$

$$\nabla_1^2 = \partial_1^2 + \partial_2^2, \qquad \dot{T} = \frac{\partial T}{\partial t}$$

Here $\varkappa = \lambda/\varrho c$ where λ is the conductivity coefficient, c the specific heat and ϱ is the density. Furthermore, $Q = W/\varrho c$ where W is the amount of heat generated per unit time and volume by the heat source.

θ denotes the temperature of the surrounding medium and $\varepsilon = 2\lambda_1/\lambda h$ where λ_1 is the external conductivity coefficient. The equation 2.1 concerns the case of heat exchange on the planes $x_3 = \pm h/2$ bounding the disc. If the disc is thermally insulated on these planes it should be assumed that $\varepsilon = 0$ in the equation (2.1).

Consider first a disc of perfectly elastic material. Expressing the stresses $\sigma_{ij}^{(0)}$ due to the temperature field by means of the derivatives of the Airy function F

$$\sigma_{ij}^{(0)} = (\nabla_1^2\delta_{ij} - \partial_i\partial_j)F \qquad i,j = 1,2 \tag{2.2}$$

and introducing these expressions into the compatibility conditions we obtain the following differential equation for the function F,[12]:

$$\nabla_1^4 F + \alpha_t E_0 \nabla_1^2 T = 0 \tag{2.3}$$

E_0 denotes here the elasticity modulus and α_t the coefficient of linear thermal expansion. If the boundary of the disc is free of tractions the boundary conditions for the equation (2.3) have the form $F = F_n = 0$ where F_n denotes the normal derivative of the function F.

Let us observe that solution of the equation (2.3) can be represented in the form

$$F(x_1, x_2, t) = \alpha_t E_0 \int\int_{(\Gamma)} T(\xi_1, \xi_2, t)\nabla^2 F^*(x_1, x_2, \xi_1, \xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \tag{2.5}$$

where the Green function F^* satisfies the differential equation

$$\nabla_1^4 F^* + \delta(x_1 - \xi_1)\,\delta(x_2 - \xi_2) = 0 \tag{2.6}$$

and the same boundary conditions as the function F. For a multiple-connected disc with the contours $s_1, s_2, \ldots s_n$, it is convenient to use the equation (2.5) in the form

$$F = \alpha_t E_0 \Big\{ \iint\limits_{(\Gamma)} F^* \nabla_1^2 T \,\mathrm{d}\xi_1 \,\mathrm{d}\xi_2 + \int\limits_{(s_1 \ldots s_n)} (TF^*_{,n} - F^* T_{,n})\,\mathrm{d}s \Big\} \tag{2.7}$$

For a simply-connected disc which is free of tractions on the boundary the curvilinear integral vanishes

$$F = \alpha_t E_0 \iint\limits_{(\Gamma)} F^* \nabla_1^2 T \,\mathrm{d}\xi_1 \,\mathrm{d}\xi_2 \tag{2.8}$$

Thus, if the temperature field is stationary and no heat sources are present inside the disc, and moreover no heat exchange occurs on the planes $x_3 = \pm h/2$, then $\nabla_1^2 T = 0$ and hence also $F = 0$ at all points of the plate. In this case therefore the disc is free of stresses. This statement constitutes the celebrated theorem of N. I. Muskhelishvili[13].

We proceed to the determination of the thermal stresses σ_{ij} due to the action of a temperature field T in a viscoelastic disc, by means of the Airy function χ. Namely we have

$$\sigma_{ij} = (\nabla_1^2 \delta_{ij} - \partial_i \partial_j)\chi \qquad i, j = 1,2 \tag{2.9}$$

Making use of the elastic-viscoelastic analogy we represent the differential equation for the Airy function in the form

$$\nabla_1^4 \bar{\chi} + \alpha_t \bar{E} \nabla_1^2 \bar{T} = 0 \tag{2.10}$$

where

$$\bar{\chi} = \int\limits_0^\infty e^{-pt} \chi(x_1, x_2, t)\,\mathrm{d}t, \qquad \bar{E} = \int\limits_0^\infty e^{-pt} E(t)\,\mathrm{d}t$$

and

$$\bar{E}(p) = \frac{\bar{\mu}(3\bar{\lambda} + 2\bar{\mu})}{\bar{\lambda} + \bar{\mu}}$$

Comparing equations (2.10) and (2.3) in which the Laplace transformation has been carried out, and assuming that the boundary conditions are homogeneous we obtain the relations

$$\bar{\chi} = \bar{f}(p) p \bar{F}; \qquad \bar{\sigma}_{ij} = \bar{f}(p) p \bar{\sigma}_{ij}^{(0)} \tag{2.11}$$

where

$$\bar{f}(p) = \frac{\bar{E}(p)}{E_0(p)}$$

Thus, we have arrived at the correspondence relation. Inverting the Laplace transformation in the relations (2.11) we have

$$\chi(x_1, x_2, t) = \int_0^t f(t-\tau)\frac{\partial}{\partial\tau} F(x_1, x_2, \tau)\,\mathrm{d}\tau \tag{2.12}$$

$$\sigma_{ij}(x_1, x_2, t) = \int_0^t f(t-\tau)\frac{\partial}{\partial\tau}\sigma_{ij}^{(0)}(x_1, x_2, \tau)\,\mathrm{d}t \qquad i,j = 1,2 \tag{2.13}$$

The problem of determination of the function χ is therefore reduced to carrying out the convolution (2.12) which contains the known function F of the corresponding perfectly elastic problem, and the function $f(t)$ describing the viscoelastic properties of the disc under consideration.

There is no difficulty in finding the function $f(t)$. Thus, for the Maxwell solid where

$$P_1(p) = \frac{1}{\vartheta} + p, \quad P_2(p) = 2\mu_0 p, \quad P_3(p) = 1, \quad P_4(p) = 3K_0$$

we have

$$f(p) = \frac{1}{p+\varkappa_1}, \quad \varkappa_1 = \frac{E_0}{3\mu_0\vartheta}, \quad f(t) = e^{-\varkappa_1 t}$$

Here $\vartheta = \eta/\mu_0$ is the time of relaxation and η is the viscosity of the material. Taking into account (2.13) and integrating by parts we obtain

$$\sigma_{ij} = \sigma_{ij}^{(0)} - \frac{1}{\varkappa_1} e^{-\varkappa_1 t}\int_0^t e^{\varkappa_1\tau}\sigma_{ij}^{(0)}(x_1, x_2, \tau)\,\mathrm{d}\tau, \qquad i,j = 1,2 \tag{2.14}$$

For the Kelvin solid

$$P_1(p) = 1, \quad P_2(p) = 2\mu_0(1+\vartheta^* p), \quad P_3(p) = 1, \quad P_4(p) = 3K_0$$

whence

$$\bar{f}(p) = \frac{1}{p} + \frac{c}{p+\varkappa_2}, \quad c = \frac{2(1+\nu_0)}{1-2\nu_0}, \quad \varkappa_2 = \frac{3}{(1-2\nu_0)\vartheta^*}$$

ϑ^* denoting the retardation time; the equation (2.15) assumes the form

$$\sigma_{ij} = \sigma_{ij}^{(0)}(1+c) - \frac{c}{\varkappa_2} e^{-\varkappa_2 t}\int_0^t e^{\varkappa_2\tau}\sigma_{ij}^{(0)}(x_1, x_2, \tau)\,\mathrm{d}\tau \tag{2.15}$$

If the temperature field is stationary the equation (2.10) takes the form

$$\nabla_1^4\bar{\chi} + \alpha_t\frac{\bar{E}}{p}\nabla^2 T = 0 \tag{2.16}$$

Comparing the latter equation with equation (2.3) we find

$$\bar{\chi} = \bar{f}(p)F \tag{2.17}$$

Hence

$$\chi(x_1, x_2, t) = f(t)F(x_1, x_2), \quad \sigma_{ij}(x_1, x_2, t) = f(t)\sigma_{ij}^{(0)}(x_1, x_2) \tag{2.17'}$$

For the Maxwell solid

$$\sigma_{ij}(x_1, x_2, t) = e^{-\varkappa_1 t}\sigma_{ij}^{(0)}(x_1, x_2) \tag{2.18}$$

while for the Kelvin solid

$$\sigma_{ij}(x_1, x_2, t) = (1+ce^{-\varkappa_2 t})\sigma_{ij}^{(0)}(x_1, x_2) \tag{2.19}$$

The determination of stresses is especially simple in the case of an infinite disc. In the disc of perfectly elastic material

$$\sigma_{ij}^{(0)} = 2\mu_0(\partial_i\partial_j - \delta_{ij}\nabla^2)\phi^{(0)} \qquad i,j = 1,2 \tag{2.20}$$

where $\phi^{(0)}$ is the thermoelectric displacement potential yielding the displacements according to the formula $u_i = \phi_{,i}^{(0)}$. The function $\phi^{(0)}$ satisfies the Poisson equation

$$\Delta_1^2\phi^{(0)} = m_0 T, \qquad m_0 = (1+\nu_0)\alpha t \tag{2.21}$$

Making use of the elastic-viscoelastic analogy we express the Laplace transformations of the stresses σ_{ij} in the viscoelastic disc by means of the relations

$$\bar{\sigma}_{ij} = 2\bar{\mu}(\partial_i\partial_j - \delta_{ij}\nabla_1^2)\bar{\phi} \qquad i,j = 1,2 \tag{2.22}$$

the function $\bar{\phi}$ being the particular solution of the equation

$$\nabla^2\bar{\phi} = \bar{m}\bar{T}, \qquad \bar{m} = (1+\bar{\nu})\alpha_t \tag{2.23}$$

A comparison of the equations (2.20) and (2.21) after applying the Laplace transform, with the equations (2.22) and (2.23) yields

$$\sigma_{ij} = 2(\partial_i\partial_j - \delta_{ij}\nabla_1^2)\psi \qquad i,j = 1,2 \tag{2.24}$$

where

$$\psi(x_1, x_2, t) = \int_0^t g(t-\tau)\frac{\partial}{\partial\tau}\phi^{(0)}(x_1, x_2, \tau)\,\mathrm{d}\tau \tag{2.25}$$

and

$$g(t) = \alpha^{-1}\bar{g}(p) \qquad \bar{g}(p) = \frac{\bar{\mu}\bar{m}}{pm_0}$$

Consider the action of an instantaneous heat source situated at the origin of the coordinate system in the viscoelastic disc. Assuming that

the disc is thermally insulated on the planes $x_3 = \pm h/2$ we have the following solution of the equation (2.1):

$$T(r,t) = \frac{Q}{4\pi\varkappa t}\exp\left[-\frac{r^2}{4\varkappa t}\right] \qquad r = (x_1^2+x_2^2)^{1/2} \tag{2.25}$$

Solving (2.21) we have $\phi^{(0)}$

$$\phi^{(0)}(r,t) = \frac{Qm_0}{2\pi}\left[\ln r - \frac{1}{2}Ei\left(-\frac{r^2}{4\varkappa t}\right)\right]$$

where

$$-Ei(-\eta) = \int_\eta^\infty \frac{e^{-u}}{u}\,du$$

Equation (2.25) yields the function

$$\psi = \frac{Q}{2\pi}\int_0^t g(t-\tau)\left[\delta(\tau)\ln r + \frac{1}{\tau}\exp\left(-\frac{r^2}{4\varkappa\tau}\right)\right]d\tau$$

It is found for the Maxwell solid that

$$\psi = \frac{Q\mu_0}{2\pi}\left[e^{-\varkappa_1 t}\ln r + \frac{1}{2}\int_0^t e^{-\varkappa_0(t-\tau)}\frac{\exp\left(-\frac{r^2}{4\varkappa\tau}\right)}{\tau}\,d\tau\right] \tag{2.26}$$

where

$$\varkappa_1 = \frac{E_0}{3\vartheta\mu_0}$$

while for the Kelvin solid

$$\psi(r,t) = \frac{Q\mu_0}{2\pi}\left[\ln r - \frac{1}{2}Ei\left(-\frac{r^2}{4\varkappa t}\right) + ce^{-\varkappa_2 t}\ln r + \right.$$
$$\left. + \frac{c}{2}\int_0^t e^{-\varkappa_0(t-\tau)}\frac{\exp\left(-\frac{r^2}{4\varkappa\tau}\right)}{\tau}\,d\tau\right] \tag{2.27}$$

$$\varkappa_2 = \frac{3}{(1-2\nu_0)\vartheta^*}, \qquad c = \frac{2(1+\nu_0)}{1-2\nu_0}$$

The function ψ being known we determine the stresses in accordance with the formulae

$$\sigma_{rr} = -2r^{-1}\psi_{,r}, \qquad \sigma_{\varphi\varphi} = -2\psi_{,rr} \tag{2.28}$$

In the case of a stationary temperature field

$$\phi = s(t)\phi^{(0)}; \quad \sigma_{ij} = f(t)\sigma_{ij}^{(0)} \tag{2.29}$$

where

$$s(t) = \alpha^{-1}\left(\frac{\overline{m}}{pm_0}\right) \qquad f(t) = \alpha^{-1}\left(\frac{\overline{E}}{pE_0}\right)$$

Let us observe that the plate analogy may be employed in the determination of the function $\overline{\chi}$ from the equation (2.10). Consider a perfectly elastic plate of the same shape as the disc under consideration. Let the plate be clamped $W = W_n = 0$ on the boundary and the loading varying in time. The differential equation for the plate deflection assumes the form after carrying out the Laplace transformation

$$\nabla_1^4 \overline{w} - \frac{\overline{g}}{N_0} = 0 \tag{2.30}$$

where N_0 is the bending rigidity of the plate. Comparing the equations (2.10) and (2.30) and assuming that the boundary of the viscoelastic disc is free of tractions ($\chi = \chi, n = 0$) we discover that when $\overline{\chi} = \overline{w}$

$$\overline{q}(x_1, x_2, p) = -\alpha_t N_0 \overline{E}(p)\nabla_1^2 \overline{T}(x_1, x_2, p) \tag{2.31}$$

In view of the analogy between the differential equations and the boundary conditions a solution of the equation (2.10) may be replaced by the solution of the problem of bending of a clamped plate loaded by the load (2.31). This analogy is successfully being applied in thermoelasticity problems[14,15].

We now proceed to the investigation of the dynamical problem of propagation of stresses in a semi-infinite disc, produced by a sudden heating of the disc boundary. We assume that the disc is thermally insulated on the planes $x_3 = \pm h/2$. The boundary condition for the temperature being $T(0, t) = T_0 H(t)$ ($H(t)$ is the Heaviside function) and the initial condition $T(x_1, 0) = 0$ the temperature field is given by the expression

$$T(x_1, t) = T_0 \operatorname{erfc} \frac{x_1}{\sqrt{4\varkappa t}}, \qquad x_1 > 0 \tag{2.32}$$

Performing the Laplace transformation in (2.32) we have

$$\overline{T}(x_1, p) = \frac{T_0}{p} \exp(-x_1\sqrt{p/\varkappa}), \qquad x_1 > 0 \tag{2.33}$$

or

$$\overline{T}(x_1, p) = \frac{2T_0}{\pi p}\int_0^\infty \frac{\alpha \sin \alpha x_1 \, d\alpha}{\alpha^2 + p/\varkappa} \tag{2.33'}$$

In the problem under investigation the equation (1.31) for the thermoelastic displacement potential is reduced to the form

$$\bar{\phi}_{,11}-p^2\bar{\sigma}^2\bar{\phi}=\bar{m}\bar{T}, \qquad \bar{m}=(1+\bar{\nu})\alpha t \tag{2.34}$$

for we are dealing with a one-dimensional plane problem. The solution of the latter equation is representable in the form

$$\bar{\phi}=-\frac{2T_0\bar{m}}{\pi p}\int_0^\infty \frac{\alpha \sin \alpha x_1 \, d\alpha}{(\alpha^2+p/\varkappa)(\alpha^2+p^2\bar{\sigma}^2)} \tag{2.35}$$

or after integration

$$\bar{\phi}=-\frac{T_0\bar{m}}{p(p^2\bar{\sigma}^2-p\varkappa^{-1})}\left[e^{-x_1\sqrt{p/\varkappa}}-e^{-x_1 p\bar{\sigma}}\right] \tag{2.36}$$

For $x_1=0$ we have $\bar{\phi}=0$ and since $\bar{\sigma}_{11}=p^2\bar{\phi}$ also $\sigma_{11}(0,p)=0$, which was to be expected.

Consider now the viscoelastic solid of M. A. Biot. We assume that $a(t)$ and $b(t)$ have the same relaxation time ε^{-1} and are expressed by the simple exponential formulae

$$a(t)=\mu_0 e^{-\varepsilon t} \qquad b(t)=\lambda_0 e^{-\varepsilon t}$$

Since

$$\bar{m}=\frac{2\bar{\mu}+3\bar{\lambda}}{2(\bar{\lambda}+\bar{\mu})}\alpha_t=m_0=\text{const} \qquad \bar{\sigma}^2=\sigma_0^2\frac{p+\varepsilon}{p} \quad \bar{\mu}=\mu_0\frac{p}{p+\varepsilon}$$

where $\sigma_0=\dfrac{p}{\lambda_0+2\mu_0}$, let us represent the function $\bar{\phi}$ in the form

$$p^2\bar{\phi}=-\frac{T_0 m_0 p^2}{\sigma_0^2(p-\beta)}\left(e^{-x_1\sqrt{p/\varkappa}}-e^{-p\sigma_0\sqrt{p(p+\varepsilon)}}\right), \qquad \beta=\frac{1}{\varkappa\sigma_0^2}-\varepsilon>0, {}_1\varepsilon>0 \tag{2.37}$$

Inverting the Laplace transformation in the equation (2.37) and introducing the notations

$$\xi=\frac{x_1}{\varkappa\sigma_0}, \qquad \tau=\frac{t}{\varkappa\sigma_0^2}, \qquad a=\varepsilon\varkappa\sigma_0^2$$

we obtain

$$\frac{\partial^2\phi}{\partial t^2}=-\frac{T_0 m_0}{\sigma_0^2}[f_1(\xi,\tau;\alpha)-g_1(\xi,\tau;\alpha)],$$

where

$$f_1(\xi,\tau;\alpha)=\frac{1}{2}e^{(\tau 1-\alpha}\left[e^{-\xi\sqrt{1-\alpha}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\sqrt{\tau(1-\alpha)}\right)+\right.$$
$$\left.+e^{\xi\sqrt{1-\alpha}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}}+\sqrt{\tau(1-\alpha)}\right)\right]$$

$$g_1(\xi,\tau;\alpha)=e^{\tau(1-\alpha)}\left[e^{-\xi\left(1-\frac{\alpha}{2}\right)}H(\tau-\xi)+\right.$$

$$\left.+\xi\frac{\alpha}{2}\int_0^\tau e^{-\eta\left(1-\frac{\alpha}{2}\right)}\frac{I_1\left(\frac{\alpha}{2}\sqrt{\eta^2-\xi^2}\right)}{\sqrt{\eta^2-\xi_2}}H(\eta-\xi)\,\mathrm{d}\eta\right]$$

The change in time of the stress $\sigma_{11}=\varrho\dfrac{\partial^2\phi}{\partial t^2}$ is described for $\tau<\xi$ by the function f_1, and for $\tau>\xi$ by the functions f_1 and g_1. The function f_1 is of diffusional nature, while g_1 is of nature of a longitudinal wave the front of which moves with the velocity $c=1/\sigma_0$. For $\tau=\xi$ the following jump in stresses occurs:

$$\sigma_{11}(\xi,\tau_-;\alpha)-\sigma_{11}(\xi,\tau_+;\alpha)=\frac{T_0m_0\varrho}{\sigma_0^2}e^{-\frac{\alpha\xi}{2}}$$

It depends on α and varies with the distance x_1.

On the boundary of the disc we have

$$\sigma_{11}=0,\quad \sigma_{22}=\sigma_{33}=-2\mu_0m_0T_0e^{-\alpha\tau}$$

As $\alpha\to 0$, i.e. for a perfectly elastic body the above solution reduces to that derived by V. I. Danilovskaya[16].

3. THERMAL STRESSES IN PLATES

Let there exist a temperature field $T=x_3\tau(x_1,x_2,t)$ in a plate, where the function τ satisfies the equation

$$\nabla^2\tau-\frac{1}{\varkappa}\dot{\tau}-\beta^2(\tau-\beta)=-q/\varkappa \tag{3.1}$$

Here $\beta^2=\dfrac{\sigma\lambda_1}{\lambda h}$, $=\Theta\dfrac{\Theta_1-\Theta_2}{h}$, Θ_1 is the temperature of the surrounding medium below the plate and Θ_2 that above the plate. The equation for the deflection of the perfectly elastic plate, due to the action of the temperature field $T=x_3T(x_1,x_2,t)$ has the form[12]

$$\nabla_1^4W+m_0\nabla_1^2\tau=0,\quad m_0=(1+\nu_0)\alpha_t \tag{3.2}$$

The bending and torsional moments are given by the formulae

$$M_{ij}=-N_0\{(1-\nu_0)w_{,ij}+\delta_{ij}[\nu_0\Delta^2w+(1+\nu_0)\alpha_t\tau]\}\qquad i,j=1,2 \tag{3.3}$$

The solution of the equation (3.2) can be represented in the form

$$w(x_1,x_2,t)=m_0\int\!\!\int_{(r)}\tau(\xi_1,\xi_2,t)\nabla^2w^*(x_1,x_2;\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \tag{3.4}$$

or else

$$w(x_1, x_2, t) = m_0 \left\{ \iint\limits_{(\Gamma)} w^* \nabla_1^2 \tau \, d\xi_1 \, d\xi_2 + \int\limits_{(s_1, \ldots, s_n)} (\tau w_{,n}^* - w^* \tau_{,n}) \, ds \right\}$$

in the case of a multiple-connected plate with contours $s_1, \ldots, s_n$. The Green function appearing in the equations (3.4) and (3.5) satisfies the equation

$$\nabla_1^4 w^* + \delta(x_1 - \xi_1) \, \delta(x_2 - \xi_2) = 0 \tag{3.6}$$

and the same boundary conditions as the function w. In the case of a simplyconnected clamped plate ($w^* = w_n^* = 0$) the curvilinear integral in the expression (3.5) drops out. In the case of a stationary temperature field and absence of heat sources and heat exchange on the planes $x_3 = \pm \frac{h}{2} (\beta = 0)$ we have $w = 0$ at all points of the plate; then

$$M_{11} = M_{22} = -N_0 m_0 \tau$$

Making use of the elastic-viscoelastic analogy we represent the equation for deflection W of the viscoelastic plate in the form

$$\nabla_1^4 \overline{W} + \overline{m} \nabla_1^2 \overline{\tau} = 0, \qquad \overline{W} = \int\limits_0^\infty e^{-pt} W \, dt \tag{3.7}$$

The transformations of the bending and torsional moments are given by the formula

$$\overline{\sigma}_{ij} = -\overline{N} \left\{ (1-\nu) \overline{W}_{ij} + [\overline{\nu} \nabla_1^2 \overline{W} + \overline{m} \overline{\tau}] \, \delta_{ij} \right\} \qquad i, j = 1,2 \tag{3.8}$$

Comparing the equations (3.2) and (3.7) we can establish the correspondence relation

$$\overline{W} = \frac{\overline{m}}{m_0} \overline{w} \tag{3.9}$$

whence it follows that

$$W(x_1, x_2, t) = \int\limits_0^t h(t-\eta) \frac{\partial}{\partial \eta} w(x_1, x_2, \eta) \, d\eta \tag{3.10}$$

Here $h(t)$ is the inverse Laplace transformation of the function $\overline{h}(p) = \dfrac{\overline{m}}{p m_0}$

Introducing the notations

$$\overline{k}(p) = \frac{\overline{E}}{p E_0}, \qquad \overline{l}(p) = \frac{\overline{E}}{p E_0} \frac{\overline{\nu}}{1-\overline{\nu}}, \qquad \overline{r}(p) = \frac{\overline{E}}{p E_0} \frac{1+\nu_0}{1-\nu} \alpha_t$$

the inverse transformation of the moments takes the form

$$M_{ij} = -N_0(1-\nu_0)\int_0^t \left\{k(t-\eta)\frac{\partial}{\partial\eta}w_{,ij}+ \right.$$

$$\left. +\left[l(t-\eta)\frac{\partial}{\partial\eta}\nabla_1^2 w+r(t-\eta)\frac{\partial\tau}{\partial\eta}\right]\right\}\mathrm{d}\eta \qquad i,j=1,2 \quad (3.11)$$

If the temperature field is stationary then $T = x_3\tau(x_1, x_2)$

$$W(x_1, x_2, t) = h(t)w(x_1, x_2), \qquad h(t) = \alpha^{-1}\left(\frac{\overline{m}}{pm_0}\right) \quad (3.12)$$

and the bending and torsional moment take the form

$$M_{ij} = -N_0(1-\nu_0)\{k(t)w_{,ij}+[l(t)\nabla^2 w+r(t)\tau]\delta_{ij}\}, \qquad i,j=1,2 \quad (3.13)$$

Consider vibrations of plate due to the temperature field $T = x_3\tau(x_1, x_2, t)$. The influence of the inertia forces will be especially great if the plate be suddenly heated. For the elastic solid the equation for the deflection has the form

$$\nabla_1^4 w+\frac{\varrho h}{N_0}\ddot{w}+m_0\nabla_1^2\tau = 0 \quad (3.14)$$

The solution of this equation may be represented as a sum of two parts, the quasi-static w_1 and w_2 accounting for the influence of the inertia forces[17]:

$$\nabla_1 w_1+m_0\nabla_1^2\tau = 0$$

$$\nabla_1^4 w_2+\frac{ph}{N_0}(\ddot{w}_2+\ddot{w}_1) = 0 \quad (3.15)$$

After applying the Laplace transform to the above equations, assuming that $w_2 = (x_1, x_2, 0) = \dot{w}_2(x_1, x_2, 0) = 0$ we obtain

$$\nabla_1^4\overline{w}_1+m_0\nabla_1^2\overline{\tau} = 0$$

$$\nabla_1^4\overline{w}_2+\frac{\varrho h}{N_0}[(\overline{w}_2p^2+\overline{w}_1p^2-pw_1(x_1, x_2, 0)-\dot{w}_1(x_1, x_2, 0)] = 0 \quad (3.16)$$

Making use of the elastic-viscoelastic analogy we obtain for the deflection $W = W_1+W_2$ of the viscoelastic plate the system of equations

$$\nabla_1^4 W_1+\overline{m}\nabla_1^2\overline{\tau} = 0$$

$$\nabla_1^4\overline{W}_2+\frac{\varrho h}{\overline{N}}[p^2\overline{W}_2+p^2\overline{W}_1-pW_1(x_1, x_2, 0)-\dot{W}_1(x_1, x_2, 0)] = 0 \quad (3.17)$$

To solve the system of equations (3.17), we may employ orthogonal and normalized system of eigenfunctions $W_{nm}(x_1, x_2)$ of the elastic plate of

the same shape and the same boundary conditions. The eigenfunctions satisfy the equation

$$\nabla_1^4 w_{nm} - \frac{\omega_{nm}^2}{N_0} \varrho h w_{nm} = 0 \tag{3.18}$$

Assuming that the solution of the first equation (3.17) may be represented by the series

$$W_1(x_1, x_2, t) = \sum_{n,m}^{\infty} A_{nm}(t)\, w_{nm}(x_1, x_2) \tag{3.19}$$

and expanding the function $\overline{W}_2(x_1,x_2,p)$ into a similar series

$$\overline{W}_2(x_1, x_2, p) = \sum_{n,m}^{\infty} \overline{B}_{nm}(p) w_{nm}(x_1, x_2) \tag{3.20}$$

we arrive at the following form of the solution of the system of equations (3.17)

$$\overline{W}(x_1, x_2, p) = \sum_{n,m}^{\infty} \frac{\overline{\gamma}\omega_{nm}^2 \overline{A}_{nm} + pA_{nm}(0) + \dot{A}_{nm}(0)}{p^2 + \overline{\gamma}\omega_{nm}^2} w_{nm}(x_1, x_2) \tag{3.21}$$

Here $\overline{\gamma} = \dfrac{\overline{N}(p)}{N_0}$. Inversion of the Laplace transformation encounters considerably difficulties, even for simple models, in view of the complicated structure of the function γ. A great simplification results if we treat the solid as incompressible ($\mu_0 = \frac{1}{2}$, $K_0 = \infty$).

4. THERMAL STRESSES IN SHELLS

We shall confine ourselves to shallow shells of double curvature, basing the theory on the equations of the engineering theory of shells of V. Z. Vlasov[18], and making use of the elastic-viscoelastic analogy

$$\nabla^4 \overline{w} - \frac{1}{N} \nabla_k^2 \overline{\varphi} = -\overline{m} \nabla^2 \overline{\tau} \tag{4.1}$$

$$\nabla_k^2 \overline{w} - \frac{1}{\overline{E}h} \nabla^4 \overline{\varphi} = a_t \nabla^2 \overline{\tau}_0 \tag{4.2}$$

We have assumed here that the temperature may be expressed by the approximate relation

$$T(\alpha_1, \alpha_2, \alpha_3, t) = \tau_0(\alpha_1, \alpha_2, t) + \alpha_3 \tau(\alpha_1, \alpha_2, \tau) \tag{4.3}$$

which means that the temperature distribution in the direction of the α_3-axis of the curvilinear coordinate system is linear. The differential operators appearing in the equations (4.1) and (4.2) have the form

$$\nabla_k^2 \overline{f} = \frac{1}{A_1 A_2} \left[\partial_1 \left(k_2 \frac{A_2}{A_1} \partial_1 \overline{f} \right) + \partial_2 \left(k_2 \frac{A_1}{A_2} \partial_2 \overline{f} \right) \right]$$

$$\nabla^2 \overline{f} = \frac{1}{A_1 A_2} \left[\partial_1 \left(\frac{A_2}{A_1} \partial_1 \overline{f} \right) + \partial_2 \left(\frac{A_1}{A_2} \partial_2 \overline{f} \right) \right], \quad \partial_1 = \frac{\partial}{\partial \alpha_1}, \quad \partial_2 = \frac{\partial}{\partial \alpha_2}$$

Here $A_1(\alpha_1,\alpha_2)$, $A_2(\alpha_1,\alpha_2)$ are the coefficients of the first fundamental quadratic form of the middle surface of the shell $k_1 = \frac{1}{R_1}$, $k_2 = \frac{1}{R_2}$ denote the curvatures of the middle surface in the directions α_1 and respectively, of the curvilinear orthogonal coordinate system. Let us observe that $\overline{N} = \frac{\overline{E}}{12(1-\overline{\nu}^2)}$ and $\overline{m} = (1+\overline{\nu})\alpha_t$ depend only on the parameter p of the Laplace transformation.

The knowledge of the functions $\overline{w}$, $\overline{\varphi}$ is sufficient for determining the transformations of the forces $\overline{N}_{ij}$ and the moments $\overline{M}_{ij}$. Thus

$$\begin{aligned}
\overline{N}_{11} &= -\frac{1}{A_2}\partial_2\left(\frac{1}{A_2}\partial_2\overline{\varphi}\right) - \frac{1}{A_1 A_2}\partial_1 A_2 \cdot \frac{1}{A_1}\partial_1\overline{\varphi} \\
\overline{N}_{22} &= -\frac{1}{A_1}\partial_1\left(\frac{1}{A_1}\partial_1\overline{\varphi}\right) - \frac{1}{A_1 A_2}\partial_2 A_1 \cdot \frac{1}{A_2}\partial_2\overline{\varphi} \qquad (4.4.)\\
\overline{N}_{12} &= \frac{1}{A_1 A_2}\left[\partial_1\partial_2\overline{\varphi} - \frac{1}{A_1}\partial_2 A_1\partial_1\overline{\varphi} - \frac{1}{A_2}\partial_1 A_2\partial_2\overline{\varphi}\right]
\end{aligned}$$

and

$$\begin{aligned}
\overline{M}_{11} &= -\overline{N}(\overline{\varkappa}_{11}+\overline{\nu}\overline{\varkappa}_{22}-\overline{m}\,\overline{\tau}) \\
\overline{M}_{22} &= -\overline{N}(\overline{\varkappa}_{22}+\overline{\nu}\overline{\varkappa}_{11}-\overline{m}\,\overline{\tau}) \qquad (4.5)\\
\overline{M}_{12} &= -\overline{N}(1-\overline{\nu})\overline{\varkappa}_{12}
\end{aligned}$$

where

$$\begin{aligned}
\overline{\varkappa}_{11} &= -\frac{1}{A_1}\partial_1\left(\frac{1}{A_1}\partial_1\overline{w}\right) - \frac{1}{A_1 A_2}\partial_2 A_1 \cdot \frac{1}{A_2}\partial_2\overline{w} \\
\overline{\varkappa}_{22} &= -\frac{1}{A_2}\partial_2\left(\frac{1}{A_2}\partial_2\overline{w}\right) - \frac{1}{A_1 A_2}\partial_1 A_2 \cdot \frac{1}{A_1}\partial_1\overline{w} \qquad (4.6)\\
\overline{\varkappa}_{12} &= -\frac{1}{A_1 A_2}\left(\partial_1\partial_2\overline{w} - \frac{1}{A_1}\partial_2 A_1\partial_1\overline{w} - \frac{1}{A_2}\partial_1 A_2\partial_2\overline{w}\right)
\end{aligned}$$

The differential equations for the perfectly elastic shell have the form

$$\nabla^4\overline{w}_0 - \frac{1}{N_0}\nabla_k^2\overline{\varphi}_0 = -m_0\nabla^2\overline{\tau} \qquad (4.7)$$

$$\nabla_k^2\overline{w}_0 + \frac{1}{E_0 h}\nabla^4\overline{\varphi}_0 = \alpha_t\nabla^2\overline{\tau}_0 \qquad (4.8)$$

Comparing the equations (4.1) and (4.2) with the equations (1.7) and (4.8) we observe that no simple correspondence relations occur for the functions $\overline{w}$, $\overline{q}$ and $\overline{w}_0$, $\overline{\varphi}_0$. In the general case we have to solve the system of equations (4.1) and (4.2) completed by the appropriate boundary conditions, applying to the solution thus derived the inverse Laplace transformation.

Consider a shallow shell of double curvature, of positive Gaussian curvature, supported on the rectangle with sides a, b. In this case we may assume that the first fundamental quadratic form is identical with that for the plane, i.e.

$$ds^2 = dx_1^2 + dx_2^2$$

Moreover, we assume that the curvatures k_1, k_2 are constant and positive in the region of the shell. Thus

$$\nabla^2 = \partial_1^2 + \partial_2^2, \qquad \nabla_k^2 = k_2 \partial_1^2 + k_1 \partial_2^2$$

Eliminating from the equations (4.1) and (4.2) first the function $\overline{\varphi}$ and then the function $\overline{w}$ we arrive at the system of aquations

$$\nabla^8 \overline{w} + \overline{\lambda}^4 \nabla_k^4 \overline{w} = \overline{\lambda}^4 \alpha_t \nabla_k^2 \nabla^2 \tau_0 - \overline{m} \nabla^6 \overline{\tau} \tag{4.9}$$

$$\nabla^8 \overline{\varphi} + \overline{\lambda}^4 \nabla_k^4 \overline{\varphi} = \overline{E} h \alpha_t \nabla^6 \tau_0 + \overline{E} h \overline{m} \nabla_k^2 \nabla^2 \overline{\tau} \tag{4.10}$$

$$\overline{\lambda}^4 = \frac{12(1-\overline{\nu}^2)}{h^2}$$

Introduce the auxiliary functions $\overline{\varphi}^*$ and $\overline{w}^*$ which satisfy the differential equation

$$(\nabla^8 + \overline{\lambda}^4 \nabla_k^4)(\overline{\varphi}^*, \overline{w}^*) = \delta(x_1 - \xi_1)\, \delta(x_2 - \xi_2) \tag{4.11}$$

and the same boundary conditions as the functions $\overline{\varphi}$ and $\overline{w}$, respectively. The solution of the equations (4.9), (4.10) may be represented with the help of the appropriate Green functions in the form

$$\overline{w}(x_1, x_2, p) = \iint_{(F)} [\overline{\lambda}^4 \alpha_t \overline{\tau}_0(\xi_1, \xi_2, p) \nabla_k^2 \nabla^2 \overline{w}^*(x_1; x_2; \xi_1, \xi_2, p) - \\ - \overline{m} \overline{\tau}(\xi_1, \xi_2, p) \nabla^6 \overline{w}^*(x_1; x_2; \xi_1, \xi_2, p)]\, d\xi_1\, d\xi_2 \tag{4.12}$$

$$\overline{\varphi}(x_1, x_2, p) = \overline{E} h \iint_{(F)} [\alpha_t \overline{\tau}_0(\xi_1, \xi_2, p) \nabla^6 \overline{\varphi}^*(x_1; x_2; \xi_1, \xi_2, p) + \\ + \overline{m} \overline{\tau}(\xi_1, \xi_2, p) \nabla_k^2 \nabla^2 \overline{\varphi}^*(x_1, x_2; \xi_1, \xi_2, p)]\, d\xi_1\, d\xi_2 \tag{4.13}$$

Consider the particular case $\tau_0 = 0$, $\tau = \tau(t)$ in the whole region of the shell, assuming that the shell is simply supported. In this case the solution of equation (4.11) is furnished by the functions

$$\overline{w}^* = \overline{\varphi}^* = \frac{4}{ab} \sum_{n,m}^{\infty} \frac{\sin \alpha_n \xi_1 \sin \beta_m \xi_2}{\overline{D}_{nm}} \sin \alpha_n x_1 \sin \beta_m x_2 \tag{4.14}$$

$$\alpha_n = \frac{n\pi}{a}, \qquad \beta_m = \frac{m\pi}{b}$$

where

$$\overline{D}_{nm} = (\alpha_n^2 + \beta_m^2)^4 + \overline{\lambda}^4 (k_2 \alpha_n^2 + k_1 \beta_m^2)^2$$

Inserting (4.14) into equations (4.12) and (4.13) and integrating we obtain

$$\bar{w}(x_1, x_2, p) = -\frac{16\bar{m}\bar{\tau}}{ab} \sum_{n,m}^{\infty} \frac{(\alpha_n^2+\beta_m^2)^3}{\alpha_n\beta_m \bar{D}_{nm}} \sin\alpha_n x_1 \sin\beta_m x_2 \tag{4.15}$$

$$\bar{\varphi}(x_1, x_2, p) = \frac{16\bar{E}\bar{h}\bar{m}\bar{\tau}}{ab} \sum_{n,m}^{\infty} \frac{(\alpha_n^2+\beta_m^2)(\alpha_n^2 k_2+\beta_m^2 k_1)}{\alpha_n\beta_m \bar{D}_{nm}} \sin\alpha_n x_1 \sin\beta_m x_2 \tag{4.16}$$

$$n,m = 1,3,5 \ldots \infty$$

It can easily be verified that the boundary conditions

$$\begin{aligned} &\bar{u}_2 = \bar{w} = \bar{M}_{11} = \bar{N}_{11} = 0 \quad \text{for } x_1 = 0,a \\ &\bar{u}_1 = \bar{w} = \bar{M}_{22} = \bar{N}_{22} = 0 \quad \text{for } x_2 = 0,b \end{aligned} \tag{4.17}$$

have been satisfied. Expressing for the given model of the viscoelastic solid the quantities $\bar{m}$, $\bar{E}$, $\bar{\lambda}^4$ in an explicit form and inverting the Laplace transformation we arrive at the required functions w and φ.

A considerable simplification results in the case of an incompressible body $\left(K \to \infty \text{ or } \nu = \frac{1}{2}\right)$. In this particular case $\lambda^4 = \frac{g}{h^2}$, $\bar{m} = \frac{3}{2}\alpha_t$, $\bar{E} = \frac{3P_2(p)}{2P_1(p)}$ and $\bar{D}_{nm}$ is independent of the parameter p. We find from equations (4.9) and (4.10)

$$\bar{w} = \bar{w}_0 \quad \bar{\varphi} = \frac{3P_2(p)}{2E_0 P_1(p)} \bar{\varphi}_0 \tag{4.18}$$

where w_0, φ_0 denote the solution for the corresponding shell of perfectly elastic material, under the assumption $\nu_0 = \frac{1}{2}$.

Analogous relations (4.18) are also deduced for a shallow cylindrical or spherical shell.

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